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A Relation between the Dirac Field of the Electron and Electromagnetic Fields

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Abstract

There is a point of view from which a field governed by the Dirac equation for the electron is the same as a field governed by the Maxwell-Lorentz equations for electromagnetic fields. This observation suggests the possibility that the two sets of equations are of the same origin.

1. *Introduction*

The purpose of this paper is to demonstrate that it is sensible to speculate that the Dirac equation for the electron and the Maxwell-Lorentz equations for electromagnetic fields are derivable, in two different ways of approximation respectively, from the same one set of nonlinear equations covariant under the coordinate transformation of a non-Euclidean sense. During the last fifty years, this kind of speculation was prohibited strictly by the conventional discipline of quantum mechanics. The principle of indeterminacy and that of superposition make the notion of spinor physically significant; conversely, the apparent necessity of the notion of spinor, demonstrated first by Pauli, make those principles indispensable ones in physics. This is a situation which cannot be found in classical mechanics. Furthermore, it has been found that the Maxwell-Lorentz equations (footnote 1) can be rewritten in terms of spinors; the resultant spinor equations differ considerably from the Dirac equation (Bade and Jehle, 1953). Thus one tends to believe that the notion of spinor is fundamental in physics, and there is no kinship between the Dirac equation and the Maxwell-Lorentz equations, except that they are both spinor equations. In recent years, however, it has become evident that most of those principles which characterize quantum mechanics as distinctive from classical mechanics,

[1] Throughout this paper, from here on, we call the Dirac equation for the electron simply, the Dirac equation, and the Maxwell-Lorentz equations for electromagnetic fields as the Maxwell-Lorentz equations.

including classical field theories, are of limited validity (Koga, 1972, 1973, 1974). Furthermore, it seems to be difficult, from the physical point of view, to regard the Dirac equation as a spinor equation (Koga, 1975a, 1975b). Hence we now have no significant reason for which we refrain from reexamining the possibility of interpreting the Dirac equation in the light of our understanding of classical mechanics and classical field theories.

First, in Section 2, we shall summarize difficulties found with respect to the Dirac equation, particularly from the relativistic point of view. In Section 3, we shall see certain similarity and complementarity existing qualitatively between the Dirac equation and the Maxwell-Lorentz equations. Motivated by the result of this qualitative observation, we shall find, in Section 4, that if a field governed by the Dirac equation is seen as electromagnetic in an inertial frame of reference, the field has a magnetic moment which is the same as of the Bohr magneton. This implies that a field governed by the Dirac equation has a peculiarly ordered structure which can be introduced to a field satisfying the Maxwell-Lorentz equations either by a particular boundary condition or by a particular source, i.e., the Bohr magneton. In conclusion, we shall emhasize that the physical kinship existing between the Dirac equation and the Maxwell-Lorentz equations seems to be different from those relations which have been found thus far in terms of the mathematical notion of spinor.

2. *Difficulties of the Dirac Equation*

From the mathematical point of view, the Dirac equation may be a spinor equation and is covariant under the Lorentz transformations. From the physical point of view, however, the situation is not simple.

In the first place, the quantum-mechanical notion of the motion of a particle is not compatible with the notion of the special-relativistic covariancy of a physical law. This is due to the fact that the uncertainty principle prohibits the simultaneous determination of the velocity and the position of a particle, while that determination is necessary for the Lorentz transformation (Koga, 1975b, Appendix A).

If we, being required to do so from the physical point of view, rewrite the Dirac equation to a set of four partial differential equations by choosing a set of matrices for those pertinent symbols contained in the former, the resultant partial differential equations are not symmetric with respect to the spatial coordinates, because of the anisotropy of the spin-matrix components. We note that subscripts (1, 2, 3, 4) attached to the four wave functions are not spatial-directional indices. Hence, it is a sensible conclusion that the anisotropy of the spin structure of the electron is already embodied in the Dirac equation by means of the anisotropy of the spin-matrix components, of which the choice is not completely specified in the Dirac equation. Then, a rotation of the coordinate axes must be paired with a transformation of the spin matrices instead of the wave functions. In this way, the wave functions governed by the Dirac equation are to be regarded as scalars under rotatory transformations, including the Lorentz transformation, of the coordinate axes. If we do so and regard the Dirac equation as a set of four partial differential equations, the equations are no longer covariant under those transformations of the coordinate axes (Koga, 1975b, Appendix B) (see footnote 2).

It is difficult to evaluate physically a solution of the Dirac equation in the conventional way of quantum mechanics: As noted elsewhere (Koga, 1957a, Appendix), the principle of superposition is not valid for states satisfying the Dirac equation. As a consequence, there is no Heisenberg's equation of motion which can be derivable from the Dirac equation. Obviously the Dirac equation itself has no agency to motivate or govern a rotatory motion of the anisotropic composition of the spin matrices, and hence the anisotropic structure of the electron. We may say that the Dirac equation might not be complete for governing the behavior of the electron. The footnote given for the last paragraph is also pertinent for this paragraph.

To sum up, the significance of the Dirac equation seems to lie beyond the conventional scope of quantum mechanics.

3. *Lorentz's Speculation*

As is well known, H. A. Lorentz, in the beginning of this century, assumed that all electromagnetic phenomena were ascribed to the agency of moving electric charges, i.e., electrons. Furthermore, he speculated that the mass, energy, and momentum of the electron could be of purely electromagnetic origin (Lorentz, 1952). Stimulated by this speculation, a number of authors attempted to clarify the structure of the electron as of electromagnetic nature. Those attempts, made prior to Heisenberg's discovery of quantum mechanics, were not completely successful, and the development of quantum-mechanical view of the electron in the 1920's seemed to have led most physicists to abandon Lorentz's speculation of the electron structure as unfeasible (Møller, 1952; Sommerfeld, 1964). Indeed, it is difficult to adjust the conventional quantum-mechanical view of the electron, formed as in accordance with the principle of indeterminacy and that of superposition, to the concerned speculation of Lorentz which is entirely classical-mechanical. Rather, Farady-Maxwell-Lorentz's view of electromagnetic fields has been replaced with the concept of the photon. Any field, once being quantized, loses its connection to space coordinates. Besides, it has become known that the electromagnetic field tensor, a second-rank antisymmetric tensor, can be constructed in terms

[2] It might be overly spontaneous to recall here that Euler's equations of rotatory motion of a rigid body, where the principal axes of inertia at the center of mass are taken as forming the coordinate axes and the moment of inertia is not isotropic, are not form-invariant under a rotatory transformation of the coordinate axes. Nevertheless, one might tend to note that the arbitrariness in choosing the spin matrices in the Dirac equation is analogous to the arbitrariness in choosing the three moments of inertia about the coordinate axes in Euler's equations considered above. But it is likely that the analogy is merely incidental.

of spinors of which the interpretation is made only in quantum mechanics (Bade and Jehle, 1953) (see footnote 3).

Uncovered in recent years, however, are evidences which lead us to believe that the trend of our understanding of quantum mechanics must change: In the first place, as noted in Section 2, the principle of indeterminacy and that of superposition seem to be merely of conventional validity, and so does the theory of measurement. Secondly, the concept of the photon is considered to be not only irrational in principle (Koga, 1973), but also unnecessary in many practical cases (Scully and Sargent, 1972). Furthermore, as stated in Section 2, wave functions satisfying the Dirac equation are scalars from the physical point of view, although the concept of spinor may exist as of mathematics.

Under the circumstances, one may once more be interested in Lorentz's speculation aforementioned. Instead of following Lorentz literally, however, it may be more reasonable to speculate that an electron is a localized and self-sustained field which may be governed by a set of nonlinear partial differential equations covariant under the coordinate transformation made in a non-Euclidean sense (Einstein, 1954) (see footnote 4).

According to the above speculation, the Dirac equation and the Maxwell-Lorentz equations are to be obtained by linearizing the original set of non-linear partial differential equations in two different ways of approximation respectively. Indeed, we see that the Dirac equation and the Maxwell-Lorentz equations are mutually complementary for the electron: The behavior of a bare or mechanical electron is supposed to be governed by the Dirac equation under the influence of an electromagnetic field of which the source is the bare electron itself and which is supposed to be governed by the Maxwell-Lorentz equations. Thus the really observable electron consists of two parts, the bare core and the electromagnetic part (Heitler, 1954, chapter VI). Since the two parts constitute the really observable electron, the behavior of the real electron is in fact governed by the Dirac equation and the Maxwell-Lorentz equations considered simultaneously. But they are inseparable by any experimental method. Hence, it must be merely a matter of tentative convenience to treat an electron as if separable into two distinctive parts. Rather, the electron as a localized field may extend outwards, gradually and continuously. Indeed, if some errors are tolerated, the Maxwell-Lorentz equations and the Dirac equation are mutually substitutive: As noted earlier, the whole field of the electron may be regarded as electromagnetic to an approximation (Lorentz, 1952). Also, the same whole field may be considered to be governed by the

[3] Earlier, Sachs and Schwebel rewrote the Maxwell-Lorentz equations by introducing a set of new field variables which are complex and are related linearly to the field variables appearing in the Maxwell-Lorentz equations, and obtained spinor equations. They recognize that the linear relations between the two sets of field variables are not co-variant under the Lorentz transformation (Sachs and Schwebel, 1962). Sachs suggested that those spinor equations are more fundamental than the Maxwell-Lorentz equations (Sachs, 1971). See also the fifth item of the summary given in Section 5.

[4] It seems unfeasible to accept the so-called Mach principle which states that the mass of an electron is due to the whole mass contained in the entire universe (Klein, 1971).

Schrödinger equation or the Dirac equation to an approximation; this situation is suggested by the known fact that a stationary state (energy eigenstate) of an electron bound in an atom, being determined by the Schrödinger equation, seems to be also a stationary state of the electromagnetic field accompanying the bare or mechanical electron, for there is seen no emission or absorption of radiation, as long as the bare electron stays in the energy eigenstate.

According to the above observation, one may tend to conclude that the Dirac equation and the Maxwell-Lorentz equations are not only mutually complementary but also mutually substitutive to an approximation. If this conclusion is feasible to some extent, it will be possible to compare more directly a field governed by the Dirac equation with a field governed by the Maxwell-Lorentz equations. The comparison will be made in the next section.

4. *A Comparison between the Dirac Equation and the Maxwell-Lorentz Equations*

The Dirac equation for an electron in a vacuum is given by

$$i\hbar\,\partial\Psi/\partial t - (\boldsymbol{\alpha}\,.\,\mathbf{p} + \beta mc^2)\Psi = 0 \qquad (4.1)$$

where

$$\mathbf{p} = -i\hbar c\,\partial/\partial\mathbf{r}, \qquad \mathbf{r} = (x, y, z)$$

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad (4.2)$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We regard the Dirac equation given in the above as a set of four partial differential equations. We write

$$\Psi_j = \phi_j \exp(-imc^2 t/\hbar), \qquad j = 1, 2, 3, 4 \qquad (4.3)$$

and substitute them in (4.1), obtaining

$$\partial\phi_1/\partial t + c(\partial\phi_4/\partial x - i\,\partial\phi_4/\partial y + \partial\phi_3/\partial z) = 0 \qquad (4.4)$$

$$\partial\phi_2/\partial t + c(\partial\phi_3/\partial x + i\,\partial\phi_3/\partial y - \partial\phi_4/\partial z) = 0 \qquad (4.5)$$

$$\partial\phi_3/\partial t + c(\partial\phi_2/\partial x - i\,\partial\phi_2/\partial y + \partial\phi_1/\partial z) + (2mc^2/i\hbar)\phi_3 = 0 \qquad (4.6)$$

$$\partial\phi_4/\partial t + c(\partial\phi_1/\partial x + i\,\partial\phi_1/\partial y - \partial\phi_2/\partial z) + (2mc^2/i\hbar)\phi_4 = 0 \qquad (4.7)$$

These are reduced further to

$$\frac{\partial\phi_j}{\partial(ct)} + \frac{i\hbar}{2mc}\left(\frac{\partial^2\phi_j}{\partial(ct)^2} - \Delta\phi_j\right) = 0 \qquad j = 1, 2, 3, 4 \qquad (4.8)$$

If m, the electronic mass, is sufficiently small, equation (3.8) is the same wave equation as for electromagnetic fields. If

$$|\partial^2\phi_j/\partial(ct)^2| \ll |\Delta\phi_j| \tag{4.9}$$

then the equation is the Schrödinger equation. These relations imply that the Dirac equation possesses characteristics of the Schrödinger equation and of the Maxwell-Lorentz equations at the same time (see footnote 5).

In order to analyze (4.4)-(4.7) further, we put

$$\begin{aligned} \phi_1 &= iP_x + P_y \\ \phi_2 &= -iP_z + \xi \\ \phi_3 &= i(iQ_x + Q_y) \\ \phi_4 &= i(-iQ_z + \eta) \end{aligned} \tag{4.10}$$

and substitute these in (4.4)-(4.7), obtaining

$$\operatorname{curl} \mathbf{Q} - \partial\mathbf{P}/\partial(ct) - \operatorname{grad} \eta = 0 \tag{4.11}$$

$$\operatorname{div} \mathbf{P} + \partial\eta/\partial(ct) - (2mc/\hbar)\mathbf{Q} . \mathbf{I} = 0 \tag{4.12}$$

$$\operatorname{curl} \mathbf{P} + \partial\mathbf{Q}/\partial(ct) - \operatorname{grad} \xi + (2mc/\hbar)(\mathbf{I} \times \mathbf{Q} + \eta\mathbf{I}) = 0 \tag{4.13}$$

$$\operatorname{div} \mathbf{Q} - \partial\xi/\partial(ct) = 0 \tag{4.14}$$

where

$$\begin{aligned} &\mathbf{P} = (P_x, P_y, P_z), \qquad \mathbf{Q} = (Q_x, Q_y, Q_z) \\ &\mathbf{I} = (0, 0, 1) \end{aligned} \tag{4.15}$$

Those functions are presented as if they are 3-dimensional (spatial) vectors, simply for the sake of tentative convenience. Their transformation characteristics should be reexamined in detail later. If Ψ^* is conjugate to Ψ, it is easily shown that

$$\Psi^*\Psi = \phi^*\phi = P^2 + Q^2 + \xi^2 + \eta^2 \tag{4.16}$$

$$c\Psi^*\boldsymbol{\alpha}\Psi = c\phi^*\boldsymbol{\alpha}\phi = 2c\mathbf{P} \times \mathbf{Q} + 2c(\eta\mathbf{P} - \xi\mathbf{Q}) \tag{4.17}$$

$$\Psi^*\beta\Psi = \phi^*\beta\phi = P^2 - Q^2 + \xi^2 - \eta^2 \tag{4.18}$$

If **P** denotes the electric field vector and **Q** the magnetic field vector, then functions (4.16) and (4.17) remind us of the energy-momentum density of the electromagnetic field. Function (4.18) is similar to a scalar function of the same field, known as a constituent of Mie's World function (Sommerfeld, 1964). As early investigators did, we tend to think that the structure of the electron is related to electromagnetic fields.

[5] If we substitute (4.3) in the Klein-Gordon equation, we get (4.8) immediately. Of course, solutions of (4.8) are not always solutions of (4.4)-(4.7), although the latter always satisfy (4.8).

Employing Heaviside's units, the Maxwell-Lorentz equations in a vacuum are given by

$$\text{curl}\,\mathbf{H} - \partial\mathbf{E}/\partial(ct) - \mathbf{J}/c = 0 \tag{4.19}$$

$$\text{div}\,\mathbf{E} - \rho = 0 \tag{4.20}$$

$$\text{curl}\,\mathbf{E} + \partial\mathbf{H}/\partial(ct) = 0 \tag{4.21}$$

$$\text{div}\,\mathbf{H} = 0 \tag{4.22}$$

where **E** is the electric field vector, **H** the magnetic field vector, ρ the charge density, and $\mathbf{J}/c$ the current density. ρ and **J** are assumed to be functions of **E** and **H** so that those equations are closed by themselves (Møller, 1952, chapter V). The set of equations (4.19)-(4.22) is equivalent to the set of equations (4.11)-(4.14), if we put

$$\mathbf{P} = \mathbf{E} \tag{4.23}$$

$$\mathbf{Q} = \mathbf{H} \tag{4.24}$$

$$\text{grad}\,\eta - \mathbf{J}/c = 0 \tag{4.25}$$

$$-(2mc/\hbar)\mathbf{H}\,.\,\mathbf{I} + \partial\eta/\partial(ct) + \rho = 0 \tag{4.26}$$

$$(2mc/\hbar)(\mathbf{I} \times \mathbf{H} + \eta I) - \text{grad}\,\xi = 0 \tag{4.27}$$

$$\partial\xi/\partial(ct) = 0 \tag{4.28}$$

Condition (4.28) implies that the comparison between the Dirac equation and the Maxwell-Lorentz equations should be made with respect to their time-independent solutions. Then we have from (4.26)

$$\rho = (2mc/\hbar)H_z \tag{4.29}$$

This relation leads to the following:

$$\iiint \rho\, dx\, dy\, dz = (2mc/\hbar)\iiint H_z\, dx\, dy\, dz$$

$$= (2mc/\hbar)M_z \tag{4.30}$$

where M_z is the z-component of the moment of the magnetic field. By putting

$$e = \iiint \rho\, dx\, dy\, dz \tag{4.31}$$

we obtain

$$M_z = (\hbar/2mc)e \tag{4.32}$$

The last relation is well known as of the Bohr magneton.

Equations (4.25)-(4.28) are covariant with respect to a rotation of the spatial axes, if we regard **P**, **Q** and **I** as three 3-vectors respectively, and ξ and

η as two scalars. This transformation is achieved by keeping the α's unchanged and by transforming **P** and **Q** as two 3-vectors, and ξ and η as scalars in the Dirac equation. By this transformation, of course, the resultant equation is no longer the Dirac equation. The present scheme of transformation is different either from the conventional scheme or from the one considered elsewhere (Koga, 1975b, Appendix B). The field under consideration is assumed to satisfy the Dirac equation before the transformation. But, after a rotation of the coordinate axes, the equation which the field satisfies is no longer the Dirac equation. With respect to the Lorentz transformation, the situation is worse, for equations (4.25)-(4.28) are not covariant in any conceivable way. Nevertheless, it is remarkable that, in an inertial coordinate system, a field satisfying the Dirac equation can be interpreted as to be equivalent to an electromagnetic field of which the moment of magnetic field is of the Bohr magneton.

The above situation sheds some light on the cause of difficulties which earlier investigators such as Abraham, Mie, and others encountered in their attempts to mold the structure of the electron with electromagnetic fields (Sommerfeld, 1964). Neither an electromagnetic field nor a Dirac field satisfying the Dirac equation may fully represent the real field of the electron; what is emphasized in one representation seems to be different from what is emphasized in the other.

5. *Summary and Remarks*

[1] Equation (4.8) suggests that the Dirac equation is led to Schrödinger's wave equation to an approximation, and also to the wave equations for electromagnetic fields to another approximation.

[2] Functions (4.16)-(4.18) remind us of some of the functions which played significant roles in those treatments made by Lorentz and others in the attempt to clarify the structure of the electron as of electromagnetic origin.

[3] Those relations found between the Dirac equation and the Maxwell-Lorentz equations are not covariant in any way under rotatory transformations, including the Lorentz transformation, of the coordinate axes. The cause of the situation is obvious: The field satisfying the Dirac equation is not the same as the one satisfying the Maxwell-Lorentz equations; the similarity arises only when the two fields are compared in a particular way.

[4] Relation (4.32) implies that a field governed by the Dirac equation has a peculiar regularity or order which can be introduced to a field satisfying the Maxwell-Lorentz equations either by a particular boundary condition or by a particular source, i.e., the Bohr magneton.

[5] Due to the presence of relation (4.3), the present relation between the Dirac equation and the Maxwell-Lorentz equations is different from the one considered by Sachs and Schwebel (see footnote 3). The present one is also different from those which may arise in theories developed by means of conventional spinor representations of electromagnetic fields (Bade and Jehle, 1953).

As is well known, two fields which are governed by the same set of differential equations are not necessarily of the same nature. For instance, basic equations of hydrodynamics are often similar to basic equations of the theory of electromagnetic fields. From this point of view, we are particularly interested in the fourth item in the above summary, and tend to speculate that the Dirac equation and the Maxwell-Lorentz equations are for the same one field. If it is feasible to speculate that the original equations of the field are given as covariant under coordinate transformations made in a non-Euclidean sense, and that the metric tensor components are functions of the field variables, the original equations are nonlinear. The linearization leading to the Dirac equation may be achieved by replacing certain functions with Planck's constant h and the electronic mass m, while the linearization to the Maxwell-Lorentz equations may be made by replacing a function with the electronic charge e. A preliminary investigation has been made according to this speculation (Koga, 1975c).

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